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Majority rule on heterogeneous networks

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Abstract
We focus on the majority rule (MR) applied on heterogeneous networks. When the underlying topology is homogeneous, the system is shown to exhibit a transition from an ordered regime to a disordered regime when the noise is increased. When the network exhibits modular structures, in contrast, the system may also exhibit an asymmetric regime, where the nodes in each community reach an opposite average opinion. Finally, the node degree heterogeneity is shown to play an important role by displacing the location of the order–disorder transition and by making the system exhibit non-equipartition of the average spin.

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1. Introduction

It is more and more common, nowadays, to use models and tools from statistical physics in order to describe the emergence of collective phenomena in social systems. One may think of opinion formation [1], the propagation of rumours or diseases [2], language dynamics [3], etc. Many approaches consist of representing individuals as nodes of a social network whose links represent their relations (e.g. friendship, co-authorship). The state of the node is characterized by a finite number of available states, e.g. two states—spin up and spin down, and it is usually assumed that only neighbouring agents (on the social networks) interact with each others. It is well known, on the other hand, that social networks exhibit non-trivial structures such as the small-world property [4], fat-tailed degree distributions [5, 6], a high clustering coefficient [4] and the presence of communities [7, 8]. A primordial problem is therefore to understand how this underlying topology influences the way the interactions between individuals (physical contact, discussions) may (or not) lead to collective phenomena, e.g. the outbreak of an epidemic or of a new trend/fashion.

In this paper, we address such a problem by applying the well-known majority rule (MR) [1, 9] on different topologies. MR is a very general model for opinion-formation, i.e. nodes copy the behaviour of their neighbour, thereby suggesting that the results derived in this paper should also apply to other models of the same family. In section 2, we will derive results in the mean-field approximation and show that the system undergoes a transition from a disordered phase to an ordered phase for weak enough random effects ($\sim$ low temperature). In section 3, MR is applied to a simple model for networks with two communities, the coupled random networks (CRN). It is shown that the network modularity may lead to an asymmetric regime, where the nodes in each community reach an opposite average opinion. In section 4, we introduce *dichotomous networks*, in which there are two sorts of nodes, each sort being characterized by a degree k_1 or k_2. It is shown that the location of this transition depends on the degree heterogeneity $\gamma \equiv k_2/k_1$. Moreover, the system exhibits non-equipartition of the average *magnetization* (each sort of nodes is characterized by a different average opinion/spin) when $\gamma \neq 1$. Finally, in section 5, we conclude and propose generalizations that could be of interest.

2. Majority rule

The network is composed of N nodes, each of them endowed with an opinion that can be either α or β. At each time step, one of the nodes is randomly selected and two processes may take place. With probability q, the selected node randomly picks an opinion α or β, whatever its previous opinion or the opinion of its neighbours. With probability $1-q$, two neighbouring nodes of the selected node are also selected and the three agents in this *majority triplet* all adopt the state of the local majority. The parameter q therefore measures the competition between individual choices, that have a tendency to randomize the opinions in the system, and neighbouring interactions, that tend to homogenize the opinions of agents. In the case $q = 0$, it is well known that the system asymptotically reaches global consensus where all nodes share the same opinion [9]. In the other limiting case $q = 1$, the system is purely random and the average (over the realizations of the random process) number of nodes with opinion α at time t, denoted by $\langle N_{\alpha;t}\rangle = A_t$, goes to $N/2$.

In a network of individuals that are highly connected (in order to justify the use of mean-field methods) and where all the nodes are equivalent (homogeneous network), it is straightforward to show [10] that the mean-field rate equation for A_t reads

$$A_{t+1} = A_t + q\left(\tfrac{1}{2} - a_t\right) - 3(1-q)a_t\left(1 - 3a_t + 2a_t^2\right), \qquad (1)$$

where $a_t = A_t/N$ is the average proportion of nodes with opinion α. The term proportional to q accounts for the individual random flips while the second term accounts for majority processes. One should stress that equation (1) is obtained by assuming that $\langle N^3_{\alpha;t}\rangle \approx A_t^3$, which is certainly not valid in finite size systems. An analytical description, by means of Fokker–Planck equations for instance, would therefore be of interest in that case.

It is easy to show that $a = 1/2$ is always a stationary solution of equation (1), as expected from symmetry reasons. $a = 1/2$ corresponds to a disordered state where no collective opinion has emerged in the system. However, this solution $a = 1/2$ ceases to be stable when $q < 3/5$. In that case, the system reaches one of the following ordered solutions:

$$a_- = \frac{1}{2} - \sqrt{\frac{3-5q}{12(1-q)}}, \qquad a_+ = \frac{1}{2} + \sqrt{\frac{3-5q}{12(1-q)}}. \qquad (2)$$

The system therefore undergoes an order–disorder transition at $q = 3/5$. Under this value, a collective opinion has emerged due to the *imitation* between neighbouring nodes. In the limit case $q \to 0$, one finds $a_- = 0$ and $a_+ = 1$ in agreement with the results of [9].

3. Coupled random networks

Let us now apply MR on a network with modular structures [11, 12], i.e. composed of highly connected communities, while nodes in different communities are sparsely connected. To do so, we focus on a simple model for networks with two communities, the coupled random networks. CRN is obtained by considering a set of N nodes that one divides into two classes, 1 and 2, and by randomly assigning links between the nodes. The probability for two nodes to receive a link is p_{cross} if they belong to different classes and p_{in} otherwise. The inter-connectivity between the communities is therefore tunable through the parameter $\nu = p_{\text{cross}}/p_{\text{in}}$. In the following, we focus on the interval $\nu \in [0, 1]$ for which inter-community links are less frequent than intra-community links.

It is possible [12] to derive master equations for A_1 and A_2, that are the average number of nodes of types 1 and 2 with opinion α. By doing so, one can show that the system may either reach a disordered state $A_1 = A_2 = N/2$ when $q > 3/5$ or a symmetric ordered state $\frac{N}{2} \pm N\sqrt{\frac{3-5q}{12(1-q)}}$ when $q < 3/5$. The main difference from the homogeneous case is that a new regime (figure 1) may take place when the inter-connectivity ν is below the line

$$\nu_c(q) = -1 + 2\sqrt{\frac{3q-3}{7q-9}}. \tag{3}$$

Under this line, the system may reach an asymmetric state where $|a_1 - a_2| \neq 0$, namely $a_1 = 1/2 + \delta_A$, $a_2 = 1/2 - \delta_A$ where

$$\delta_A^2 = \frac{3 - 2\frac{q(1+\nu)^2}{1-q} - 5(\nu^2 + 2\nu)}{12 - 4(\nu^2 + 2\nu)}. \tag{4}$$

In that case, the system exhibits the coexistence of different opinions, i.e. due to the lack of interaction between communities, nodes belonging to different communities exhibit different behaviours. Interestingly, the transition from the asymmetric state to the symmetric state is discontinuous, thereby suggesting that coexistence is a very fragile state and that fluctuations in the structure of the network might be sufficient in order to destabilize it.

Let us emphasize that such asymmetric states have been observed in many situations. Amongst others, one may think of the Political Blogosphere [13], where it was observed that bloggers having a different political opinion are segregated, the existence of niche markets [14], where small communities may use products that are different from those used by the majority, language dynamics [15], where it is well known that natural frontiers may also coincide with a linguistic frontier, etc. Before going further, one should also stress that such a coexistence of opinions is not a particular feature of the majority rule. Indeed, it is easy to show by using the same formalism that coexistence also takes place for other models whose dynamics favours consensus, such as non-conservative voters [16] for instance, and that the transition from the asymmetric to the symmetric state is also discontinuous in that case.

4. Dichotomous networks

Let us now focus on the role played by the degree distribution of the underlying network. To do so, we generalize homogeneous networks in the most natural way by considering random networks whose nodes may be divided into two classes, the nodes in different classes being characterized by a different degree, k_1 or k_2. This binary mixture that we call a *dichotomous network* is particularly suitable in order to reveal the role of degree distribution. Indeed, the

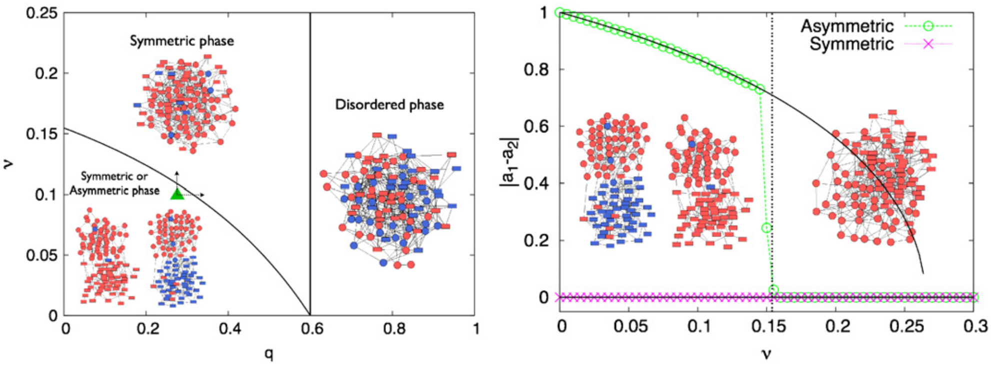

Figure 1. (Left) Phase diagram of MR on CRN. Three phases may take place: (i) a disordered phase when $q > 3/5$; (ii) a symmetric phase when $q < 3/5$; (iii) an asymmetric phase when $q < 3/5$ and when $\nu < -1 + 2\sqrt{\frac{3q-3}{7q-9}}$. (Right) Bifurcation diagram of $|a_1 - a_2|$ as a function of ν, for simulations starting from asymmetric initial conditions, when $q = 0$. The system ceases to be asymmetric above $\nu_c \approx 0.15$. The computer simulations are performed on coupled random networks with $N = 10^4$ nodes and $p_{\text{in}} = 0.01$. The simulations are stopped after 10^3 steps/node and the results averaged over 100 realizations of the random process.

degree heterogeneity is tunable through the parameter $\gamma = k_2/k_1$. When $\gamma \to 1$, one recovers a homogeneous network.

It is again possible [10] to derive equations of evolution for A_1 and A_2 and to show that the system exhibits an order–disorder transition at

$$q_c(\gamma) = \frac{8 - 16\gamma + 8\gamma^2 + 16\sqrt{K}}{24 + 16\gamma + 24\gamma^2 + 16\sqrt{K}} \tag{5}$$

where $K = 2 + 8\gamma + 16\gamma^2 + 8\gamma^3 + 2\gamma^4$. In the limiting case $\gamma \to 1$ (homogeneous network), one recovers the known result $q_c = 3/5$. It is also possible to verify that this relation is symmetric under the changes $\gamma \leftrightarrow \gamma^{-1}$, i.e. under an exchange of nodes 1 and 2. Moreover, the maximum value is obtained for $\gamma \to 0$ and $\gamma \to \infty$, $q_c(0) = q_c(\infty) = (1+2\sqrt{2})/(3+2\sqrt{2})$. Our first conclusion is therefore that the location of the order–disorder transition depends in a non-trivial way on the degree heterogeneity γ. Moreover, the location of the transition is shifted to higher values of q when the system is more heterogeneous and $q_c(1) \leqslant q_c(\alpha)$. This shift might have striking consequences in realistic situations as implies that a change of the underlying topology might lead to a transition and to the emergence of a new (ordered or disordered) phase in the network.

Let us now focus on the behaviour of a_1 and a_2 below q_c. It appears (see figure 2) that a_1 and a_2 reach different asymptotic values $a_{1,\infty} \neq a_{2,\infty}$ and that the class of nodes with the higher degree exhibit larger deviations to 1/2 than the other class. This may be understood by the fact that nodes with a higher degree are more often selected in majority triplets, thereby triggering their tendency to reach consensus. This non-equipartition of opinion is a striking feature that implies that the state of a social agent strongly depends on its connectivity. One expects that this effect should also take place in other models for opinion formation, at least if the frequency of interaction between agents is heterogeneous. Non-equipartition of opinion could be searched in empirical data, in the Blogosphere for instance, by averaging the state of agents having the same connectivity and looking for a relation between this average value and the connectivity.

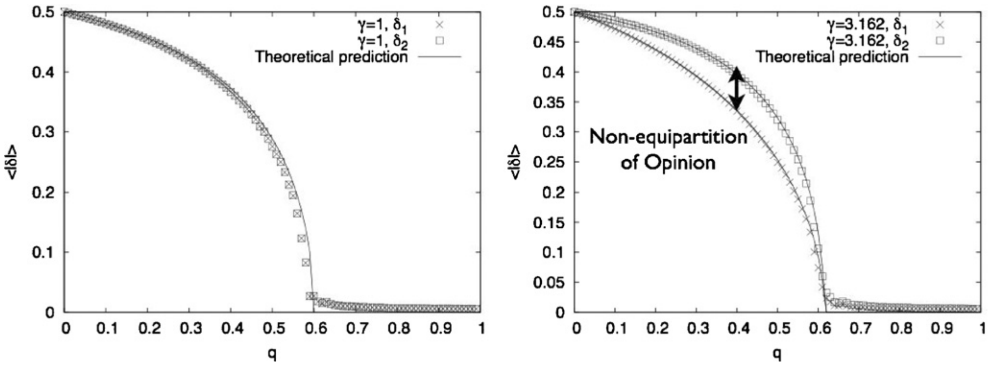

Figure 2. Bifurcation diagram of $\delta_i(q) \equiv a_i(q) - 1/2$ for $\gamma = 1$ (left) and $\gamma = 3.162$ (right), for a system composed of $N = 10^4$ nodes. Solid lines are the theoretical predictions. When $\gamma = 3.162$, one observes that q_c is slightly above 3/5. The computer simulations are performed with $N = 10^4$ nodes. The average opinions are measured after 100 time steps/node and averaged over 100 realizations of the random process.

5. Perspectives

To conclude, we would like to point to possible generalizations that would make the model more realistic. In this paper, we have focused on topologies that are composed of only two sets of nodes. A generalization of our analytical approach to more than $K > 2$ sets is in principle straightforward, but could be lengthy as it requires the solution of a set of K coupled nonlinear equations. In the case of K communities, one expects similar behaviours as those described in this paper, i.e. the emergence of asymmetric states. In that case, however, it could be interesting to study multi-opinion Potts-like models in order to reveal the richness of the network structure [17]. When the sets of nodes are characterized by different degrees k_i, it would also be interesting to generalize the prediction for the location of the transition q_c and, ultimately, to derive a prediction for q_c as a function of the degree distribution of the network.

Let us also stress that we have only focused on static topologies in this paper: the interaction pattern is fixed and only opinions, not connections are allowed to change. The opposite case is often considered in studies of network formation, i.e. nodes are endowed with fixed attributes and links are created depending on these node properties [18]. Real systems, however, are mostly in between these two extreme cases [19, 20], as intrinsic properties of nodes (like opinions) and connections among them both vary in time, sometimes over comparable temporal scales. It would therefore be interesting to study the interplay between MR and link redirection, e.g. links between nodes with different opinions could be removed, in order to answer the questions: does coevolution also lead to the coexistence of different opinions? What are the topological properties of the generated networks?

Acknowledgments

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